

Introduction to Statistical Learning

Final exam (3 pages)

Duration : 2h00 - Lecture notes allowed

Notations

- **Indicator function.** The indicator function $\mathbb{I}\{\Omega\}$ takes the value 1 if Ω is true, and 0 otherwise.
- **Empirical Rademacher average.** Consider an IID sample $Z_1^n = (Z_1, \dots, Z_n)$ and let $\sigma_1, \dots, \sigma_n$ an IID sample of Rademacher random variables ($\mathbb{P}\{\sigma_1 = +1\} = \mathbb{P}\{\sigma_1 = -1\} = 1/2$) independent of Z_1^n . Given a class $\mathcal{T}$ of functions, we denote its empirical Rademacher average by :

$$\hat{R}_n(\mathcal{T}) = \mathbb{E} \left(\sup_{t \in \mathcal{T}} \frac{1}{n} \sum_{i=1}^n \sigma_i t(Z_i) \mid Z_1^n \right)$$

- **Kernel function - definitions and properties.** Let $k : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a positive definite and symmetric kernel function. We recall that k has the property that there exist : (i) a Hilbert space $\mathcal{H}$ equipped with scalar product $\langle \cdot, \cdot \rangle_k$ and norm $\|\cdot\|_k$ and (ii) a feature mapping $\Phi : \mathbb{R}^d \rightarrow \mathcal{H}$ such that $k(x, x') = \langle \Phi(x), \Phi(x') \rangle_k$ and $k(x, x) = \|\Phi(x)\|_k^2$ for any x, x' . Given a sample $X_1, \dots, X_n$, we denote by $K = (k(X_i, X_j))_{1 \leq i, j \leq n}$ the Gram matrix induced by the kernel function k .

Exercise 1 - Consider an IID sample $X_1, \dots, X_n$ of random vectors in $\mathbb{R}^d$.

1. Consider $\mathcal{G}$ a class of functions with values in $\{-1, +1\}$ and its empirical Rademacher average $\hat{R}_n(\mathcal{G})$, and let $\mathcal{L}$ the class of classification loss functions :

$$\mathcal{L} = \{(x, y) \mapsto \mathbb{I}\{g(x) \neq y\} : g \in \mathcal{G}\}.$$

Assume an IID sample of pairs $(X_1, Y_1), \dots, (X_n, Y_n)$ is available. What is the relation between $\hat{R}_n(\mathcal{L})$ and $\hat{R}_n(\mathcal{G})$? Provide the proof of this relation.

2. Consider the class of linear functions $\mathcal{F}_{M_2} = \{x \mapsto w^T x : w \in \mathbb{R}^d, \|w\|_2 \leq M_2\}$ and find an upper bound for the empirical Rademacher average $\hat{R}_n(\mathcal{F}_{M_2})$ in terms of M_2 , n , and $\sum_{i=1}^n \|X_i\|_2^2$.
3. Consider the class of linear functions $\mathcal{F}_{M_1} = \{x \mapsto w^T x : w \in \mathbb{R}^d, \|w\|_1 \leq M_1\}$ and assume that, for any i , we have $\|X_i\|_\infty \leq r$ almost surely. Find an upper bound for the empirical Rademacher average $\hat{R}_n(\mathcal{F}_{M_1})$ in terms of M_1 , n , r and d .
4. Consider a kernel function k and the class of functions $\mathcal{F}_M = \{x \mapsto \langle w, \Phi(x) \rangle : w \in \mathcal{H}, \|w\|_k \leq M\}$, and find an upper bound for $\hat{R}_n(\mathcal{F}_M)$ which depends on M , n , and k . Provide a simple condition on the kernel k such that the behavior of $\hat{R}_n(\mathcal{F}_M)$ as a function of n is at most $O(n^{-1/2})$.

Exercise 2 - Consider an IID sample of Rademacher random variables ($\mathbb{P}\{\sigma_1 = +1\} = \mathbb{P}\{\sigma_1 = -1\} = 1/2$).

1. Consider a random variable X such that $\mathbb{E}(X) = 0$ and $X \in [a, b]$ almost surely. Give a sketch of proof evoking the main arguments of the following result : for any $t > 0$, we have :

$$\mathbb{E}(e^{tX}) \leq e^{t^2(b-a)^2/8}$$

2. Consider $Q \subset \mathbb{R}^k$ a finite set of points. We assume that they are all contained in the Euclidean ball with center the origin and radius R . Then show that : for any $t > 0$

$$\mathbb{E} \left(\sup_{q=(q_1, \dots, q_k) \in Q} \sum_{i=1}^k \sigma_i q_i \right) \leq \frac{tR^2}{2} + \frac{\log |Q|}{t}$$

where $|Q|$ is the number of points in Q .

3. Provide the optimal choice of t in the previous question and give the expression of the optimal bound.
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Exercise 3 - Consider the following :

- $D_n = ((X_1, Y_1), \dots, (X_n, Y_n))$ an IID sample of supervised training data over $\mathcal{X} \times \mathcal{Y}$,
- $\mathcal{F}$ a class of predictors from $\mathcal{X}$ to $\mathcal{Y}$,
- $A : D_n \mapsto \hat{f}_n \in \mathcal{F}$ a learning algorithm,
- $\ell : \mathcal{Y}^2 \rightarrow \mathbb{R}_+$ a cost function such that $\ell(y, y') \leq \Lambda$ for any $y, y' \in \mathcal{Y}$, with $\Lambda > 0$,
- $L(\hat{f}) = \mathbb{E}(\ell(Y, \hat{f}(X)) \mid D_n)$ is the risk of any data-driven predictor $\hat{f}$,
- $\hat{L}_n(f) = \frac{1}{n} \sum_{i=1}^n \ell(Y_i, f(X_i))$ is the empirical risk of any predictor $f \in \mathcal{F}$.

We consider the notation D'_n for a sample of size n which differs from D_n by a single point, and $\hat{f}'_n = A(D'_n)$. We assume that, for any n , there exists a $\beta_n \geq 0$ such that for any samples D_n and D'_n and for any pair $(x, y) \in \mathcal{X} \times \mathcal{Y}$, we have : $|\ell(y, \hat{f}_n(x)) - \ell(y, \hat{f}'_n(x))| \leq \beta_n$.

1. Find an upper bound on $|L(\hat{f}_n) - L(\hat{f}'_n)|$ depending on β_n .
2. Find an upper bound on $|\hat{L}_n(\hat{f}_n) - \hat{L}_n(\hat{f}'_n)|$ depending on β_n , Λ and n .
3. Show that the quantity $L(\hat{f}_n) - \hat{L}_n(\hat{f}_n)$ satisfies the bounded differences condition and apply a well-known concentration inequality.
4. Then, show that we have, with probability at least $1 - \delta$:

$$L(\hat{f}_n) \leq \hat{L}_n(\hat{f}_n) + \beta_n + (2n\beta_n + \Lambda) \sqrt{\frac{\log(1/\delta)}{2n}}$$

5. What would be an appropriate order of magnitude for the coefficient β_n ? Can you give examples of algorithms that would display such values for β_n ?

Exercise 4 - Consider the setup of preference learning where we observe an IID sample of triples $(X_1, X'_1, Y_1), \dots, (X_n, X'_n, Y_n)$. The probabilistic model assumes that, for each i , the triple (X_i, X'_i, Y_i) is such that X_i, X'_i are IID random vectors over $\mathbb{R}^d$ and Y_i is a random variable over $\{-1, 0, +1\}$. We define the ranking error of a preference rule $g : \mathbb{R}^d \rightarrow \{-1, 0, +1\}$ as :

$$L^R(g) = \mathbb{P}\{Y \neq 0, Y \cdot (g(X') - g(X)) \leq 0\}$$

and the empirical margin ranking error as :

$$\hat{L}_{n,\rho}^R(g) = \frac{1}{n} \sum_{i=1}^n \varphi_\rho(Y_i \cdot (g(X'_i) - g(X_i))) .$$

Now consider a class $\mathcal{G}$ of preference rules and define :

$$\tilde{\mathcal{G}} = \{(x, x', y) \mapsto y(g(x') - g(x)) : g \in \mathcal{G}\} .$$

1. Provide an upper bound of the empirical Rademacher average of $\tilde{\mathcal{G}}$ in terms of the empirical Rademacher average of $\mathcal{G}$.
2. Which inequality relates the empirical Rademacher average of the loss class $\varphi_\rho \circ \tilde{\mathcal{G}}$ to the empirical Rademacher average of $\tilde{\mathcal{G}}$? Provide a proof of this inequality.
3. Show that, for any $\delta \in (0, 1)$, we have, with probability at least $1 - \delta$: for any $g \in \mathcal{G}$

$$\mathbb{E}(\varphi_\rho(y(g(x') - g(x)))) \leq \hat{L}_{n,\rho}^R(g) + c_1 \hat{R}_n(m_\rho \circ \tilde{\mathcal{G}}) + c_2(n, \delta)$$

for some c_1 and $c_2(n, \delta)$ that will have to be given explicitly.

4. Deduce from the previous question a margin error bound for $L^R(g)$ that holds with large probability for any $g \in \mathcal{G}$ and which involves the empirical ranking error of g over the sample and the complexity of $\mathcal{G}$.
 5. Specify the previous result to the case of a kernel class of functions with $\mathcal{G} = \mathcal{F}_M$ as defined in **Exercise 1**.
 6. Which algorithms can be justified by the inequalities obtained in the two previous questions.
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