

# Introduction to Statistical Learning

## Exercise sheet n°1

**Exercise 1** - Consider the binary classification model where the random pair  $(X, Y)$  has distribution  $P$  over  $\mathbb{R} \times \{0, 1\}$  and :

$$\begin{aligned}\mathcal{L}(X | Y = 0) &= \mathcal{U}([0, \theta]) \\ \mathcal{L}(X | Y = 1) &= \mathcal{U}([0, 1]) \\ p &= \mathbb{P}(Y = 1)\end{aligned}$$

with  $p, \theta \in (0, 1)$  fixed. Compute the posterior probability  $\eta(x) = \mathbb{P}(Y = 1 | X = x)$ , for any  $x \in \mathbb{R}$ , as a function of  $p, \theta$ . What if  $\theta = 1/2$ ?

**Exercise 2** - Consider the binary classification model where the random pair  $(X, Y)$  has distribution  $P$  over  $\mathbb{R}_+ \times \{0, 1\}$  and :

- the marginal distribution of  $X$  over  $\mathbb{R}_+$  is denoted  $P_X$
- the conditional distribution of  $Y$  given  $X = x$  is a Bernoulli distribution with parameter  $\eta(x) = \frac{x}{x + \theta}$ , for any  $x \in \mathbb{R}_+$ , and for fixed  $\theta > 0$ .

Find the Bayes classifier for this model (i.e. the minimizer of  $L(g) = \mathbb{P}(Y \neq g(X))$  over all measurable classifiers  $g : \mathbb{R}_+ \rightarrow \{0, 1\}$ ). Give the expression of the Bayes error  $L^* = L(g^*)$  in the case where  $P_X = \mathcal{U}([0, \alpha\theta])$  with  $\alpha > 1$ . What is the value of  $\alpha$  that maximizes  $L^*$ ?

**Exercise 3** - Let  $X = (T, U, V)^T$  où  $T, U, V$  IID real-valued random variables with exponential distribution  $\mathcal{E}(1)$ . Define  $Y = \mathbb{I}\{T + U + V < \theta\}$  with fixed  $\theta > 0$ .

1. Find the Bayes classifier  $g^*(T, U)$  when  $V$  is not observed. Give the expression of the classification error of  $g^*$  (also called Bayes error). Compute it for  $\theta = 9$ .
2. Now assume that only  $T$  is observed, and address the same questions as above.
3. Propose a classifier for  $X$  when none of  $T, U, V$  are observed. What is its classification error?

**Exercise 4** - Consider  $(X, Y)$  a random pair that models classification data with labels in  $\{0, 1\}$ . Define the following classification error

$$L_\omega(g) = \mathbb{E}(2\omega(Y) \cdot \mathbb{I}\{Y \neq g(X)\})$$

where  $\omega(0) + \omega(1) = 1$ .

1. Find the optimal elements (minimizer, error) for this risk.
2. Justify the interest of such an  $L_\omega$  in practice?
3. Now consider the unit square in  $\mathbb{R}^2$ .
  - (a) Plot the curves defined by  $g \mapsto (\mathbb{P}\{g(X) = 1 \mid Y = 0\}, \mathbb{P}\{g(X) = 1 \mid Y = 1\})$  when  $g$  varies such that  $L_\omega(g) = C$  with  $C$  fixed, for different values of  $C$ .
  - (b) Same question but assuming now that  $\mathbb{P}\{g(X) = 1\} = C$  with  $C$  fixed.

**Exercise 5** - Consider  $(X, Y)$  a random pair that models classification data with labels in  $\{0, 1\}$ . We fix  $c > 0$  and we consider classifiers with reject option  $g : \mathbb{R}^d \rightarrow \{R, 0, 1\}$ , that are evaluated with the following risk functional :

$$L_R(g) = \mathbb{P}\{Y \neq g(X), g(X) \neq R\} + c\mathbb{P}\{g(X) = R\} .$$

What is the minimizing argument of  $L_R(g)$  over all possible classifiers  $g$  with reject option? Give a practical interpretation of the result.

**Exercise 6** - We consider the model for classification data where  $X$  is a random vector on  $\mathbb{R}^d$  and  $Y$  is a random variable taking values in  $\{-1, +1\}$ . We denote  $\eta(x) = \mathbb{P}\{Y = +1 \mid X = x\}$  the posterior probability. We consider the following problems for which the question is to compute the optimal decision rule  $g^*$  or  $f^*$  - please also provide the main proof arguments.

1. Criterion to minimize :  $R(g) = \mathbb{E}((Y - g(X))^2)$  where  $g : \mathbb{R}^d \rightarrow \{-1, +1\}$
2. Criterion to minimize :  $R(f) = \mathbb{E}((Y - f(X))^2)$  where  $f : \mathbb{R}^d \rightarrow \mathbb{R}$
3. Criterion to minimize :  $A(f) = \mathbb{E}(\log(1 + e^{-Yf(X)}))$  where  $f : \mathbb{R}^d \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ .

Explain why such criteria are relevant for the binary classification problem.